

# Power Flow as an Algebraic System

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## Abstract

Steady states of alternating-current (AC) circuits have been studied in considerable detail. In 1982, Baillieul and Byrnes derived an upper bound on the number of steady states in a loss-less AC circuit [IEEE TCAS, 29(11): 724–737] and conjectured that this bound holds for AC circuits in general. We prove this is indeed the case, among other results, by studying a certain multi-homogeneous structure in an algebraisation.

## 1 Introduction

For more than 60 years [37], steady states of alternating-current (AC) circuits have been studied in considerable detail. The key problem, sometimes known as the power flow or load flow problem, considers complex voltages  $V_k$  at all buses  $k$  as variables, except for one reference bus ( $k = 0$ ), where the power is supplied. When one denotes complex admittance matrix  $Y$ , complex current  $I_k$ , and complex power  $S_k$  at bus  $k$ , the steady-state equations are based on:

$$S_k = V_k I_k^* = V_k \sum_{l \in N} Y_{k,l}^* V_l^* = \sum_{l \in N} Y_{k,l}^* V_k V_l^* \quad (1)$$

where asterisk denotes complex conjugate. This captures the complex, non-convex non-linear nature [16, 38, 11, 15, 18] of any problem in the AC model. In order to obtain an algebraic system from (1), one needs to reformulate the complex conjugate. In order to do so, one may replace all  $V_k^*$  with independent variables  $U_k$ , and filter for “real” solutions where  $U_k = V_k^* = \Re V_k - \Im V_k \imath$  once the complex solutions are obtained. Thereby, we obtain a particular structure, which allows us to prove a variety of results.

In particular, the main contributions of our paper are:

- a reformulation of the steady-state equations to a multi-homogeneous algebraic system

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- analytical results on the number and structure of feasible solutions considering losses, resolving a conjecture of Baillieul and Byrnes [2], which has been open for over three decades
- empirical results for some well-known instances, including the numbers of roots, conditions for non-uniqueness of optima, and tree-width.

Our analytical results rely on the work of Morgan and Sommese [32, 36] on multi-homogeneous structures. Our empirical results rely on Bertini [6], a leading implementation of homotopy-continuation methods. As we explain in Section 8, ours is not the first algebraisation of the system (1), cf. [37, 4, 3, 2], and there is a long history [34, 14, 26, 23, 31, 29, 9, 39, 41] of the use of homotopy-continuation methods.

## 2 The Problem

In order to make the paper self-contained, we restate of the steady-state equations. Consider a circuit represented by an undirected graph  $(N, E)$ , where vertices  $n \in N$  are called buses and edges  $\{l, m\} \in E \subseteq N \times N$  are called branches, and an admittance matrix  $Y = G + B\iota \in \mathbb{C}^{|N| \times |N|}$ , where the real part of an element is called conductance  $G = (g_{lm})$  and the imaginary part susceptance  $B = (b_{lm})$ . Each bus  $k \in N$  is associated with complex voltage  $V_k = \Re V_k + \Im V_k \iota$ , complex current  $I_k = \Re I_k + \Im I_k \iota$ , and power  $S_k = P_k + Q_k \iota$  demanded or generated. Let  $0 \in N$  correspond to a reference bus with phase  $\Im V_0 = 0$  and magnitude  $|V_0|$  fixed; powers at all other buses are fixed too. (In a variety of extensions, there are other buses, denoted generators, where voltage magnitude, but not the phase and not the power is fixed.) Each branch  $(l, m) \in E$  is associated with the complex power  $S_{lm} = P_{lm} + Q_{lm} \iota$ . The key constraint linking the buses is Kirchhoff's current law, which stipulates the sum of the currents injected and withdrawn at each bus is 0. Considering the

relationship  $I = YV$ , the steady state equations hence are:

$$\begin{aligned} P_k^g &= P_k^d + \Re V_k \sum_{i=1}^n (\Re y_{ik} \Re V_i - \Im y_{ik} \Im V_i) \\ &\quad + \Im V_k \sum_{i=1}^n (\Im y_{ik} \Re V_i + \Re y_{ik} \Im V_i) \end{aligned} \quad (2)$$

$$\begin{aligned} Q_k^g &= Q_k^d + \Re V_k \sum_{i=1}^n (-\Im y_{ik} \Re V_i - \Re y_{ik} \Im V_i) \\ &\quad + \Im V_k \sum_{i=1}^n (\Re y_{ik} \Re V_i - \Im y_{ik} \Im V_i) \end{aligned} \quad (3)$$

$$P_{lm} = b_{lm} (\Re V_l \Im V_m - \Re V_m \Im V_l) \quad (4)$$

$$\begin{aligned} &\quad + g_{lm} (\Re V_l^2 + \Im V_m^2 - \Im V_l \Im V_m - \Re V_l \Re V_m) \\ Q_{lm} &= b_{lm} (\Re V_l \Im V_m - \Im V_l \Im V_m - \Re V_l^2 - \Im V_l^2) \\ &\quad + g_{lm} (\Re V_l \Im V_m - \Re V_m \Im V_l - \Re V_m \Im V_l) \\ &\quad - \frac{\bar{b}_{lm}}{2} (\Re V_l^2 + \Im V_l^2) \end{aligned} \quad (5)$$

Additionally, one can optimise a variety of objectives over the steady states. In one commonly used objective function, one approximates the costs of real power  $P_0$  generated at the reference bus 0 by a quadratic function  $f_0$ :

$$\mathbf{cost} := f_0(P_0). \quad (6)$$

(In a variety of extensions, in which there are other buses where the power is not fixed, there would be a quadratic function for each such bus and the quadratic function of power would be summed across all of these buses.) In the  $L_p$ -norm loss objective, one computes a norm of the vector  $D$  obtained by summing apparent powers  $S(u, v) + S(v, u) \forall (u, v) \in E$  for:

$$\|D\|_p = \left( \sum_{(u,v) \in E} |S(u, v) + S(v, u)|^p \right)^{1/p}. \quad (7)$$

The usual  $\|D\|_1$  is denoted **loss** below. We consider these objectives only in Section VII, while our results in Sections IV–VI apply independently of the use of any objective function whatsoever.

### 3 Definitions from Algebraic Geometry

In order to state our results, we need some definitions from algebraic geometry. While we refer the reader to [2, 27] for the basics, we present the concepts introduced in the past three decades, not yet widely covered by textbooks. For a more comprehensive treatment, please see [32, 36, 35].

Let  $n \geq 0$  be an integer and  $f(z)$  be a system of  $n$  polynomial equations in  $z \in \mathcal{C}^n$  with support  $(A_1, \dots, A_n)$ :

$$\begin{cases} f_1(z) &= \sum_{\alpha \in A_1} f_{1\alpha} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \\ &\vdots \\ f_n(z) &= \sum_{\alpha \in A_n} f_{n\alpha} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}, \end{cases} \quad (8)$$

where coefficients  $f_{i\alpha}$  are non-zero complex numbers. It is well known [12] that the polynomials define  $n$  projective hypersurfaces in a projective space  $\mathbb{CP}^n$ .

Bézout theorem states that either hypersurfaces intersect in an infinite set with some component of positive dimension, or the number of intersection points, counted with multiplicity, is equal to the product  $d_1 \cdots d_n$ , where  $d_i$  is the degree of polynomial  $i$ . We call the product  $d_1 \cdots d_n$  the usual Bézout number.

The usual Bézout number can be improved by considering:

**Definition 1** (Structure). *Any partition of the index set  $\{1, \dots, n\}$  into  $k$  sets  $I_1, \dots, I_k$  defines a structure. There,  $Z_j = \{z_i : i \in I_j\}$  is known as the group of variables for each set  $I_j$ . The associated degree  $d_{ij}$  of a polynomial  $f_i$  with respect to group  $Z_j$  is*

$$d_{ij} \stackrel{\text{def}}{=} \max_{\alpha \in A_i} \sum_{l \in I_j} \alpha_l. \quad (9)$$

We say that  $f_i$  has multi-degree  $(d_{i1}, \dots, d_{in})$ .

Whenever for some  $j$ , for all  $i$ , the same  $d_{ij}$  is attained for all  $\alpha \in A_i$ , we call the system (11) homogeneous in the group of variables  $Z_j$ . The projective space associated to the group of variables  $Z_j$  in a structure has dimension

$$a_j \stackrel{\text{def}}{=} \begin{cases} |I_j| - 1 & \text{if (11) is homogeneous in } Z_j, \text{ and} \\ |I_j| & \text{otherwise.} \end{cases} \quad (10)$$

**Definition 2** (Multi-homogeneous Bézout Number). *Assuming  $n = \sum_{j=1}^k a_j$ , the multi-homogeneous Bézout number  $\text{Béz}(A_1, \dots, A_n; I_1, \dots, I_k)$  is defined as the coefficient of the term  $\prod_{j=1}^k \zeta_j^{a_j}$ , where  $a_j$  is the associated dimension (10), within the polynomial  $\prod_{i=1}^n \sum_{j=1}^k d_{ij} \zeta_j$ , in variables  $\zeta_j, j = 1 \dots k$  with coefficients  $d_{ij}$  are the associated degrees (9); that is  $(d_{11}\zeta_1 + d_{12}\zeta_2 + \dots + d_{1k}\zeta_k) (d_{21}\zeta_1 + d_{22}\zeta_2 + \dots + d_{2k}\zeta_k) \cdots (d_{n1}\zeta_1 + d_{n2}\zeta_2 + \dots + d_{nk}\zeta_k)$ .*

Consider the example of Wampler [42] in  $x \in \mathcal{C}^3$ :

$$\begin{cases} p_1(z) &= x_1^2 + x_2 + 1, \\ p_2(z) &= x_1 x_3 + x_2 + 2, \\ p_3(z) &= x_2 x_3 + x_3 + 3, \end{cases} \quad (11)$$

with the usual Bézout number of 8. Considering the partition  $\{x_1, x_2\}, \{x_3\}$ , where  $d_{11} = 2, d_{12} = 0, d_{21} = d_{22} = d_{31} = d_{32} = 1$ . the monomial  $\zeta_1^2 \zeta_2^1$  is

to be looked up in the polynomial  $2\zeta_1(\zeta_1 + \zeta_2)^2$ . The corresponding multi-homogeneous Bézout number is hence 4 and this is the minimum across all possible structures.

In general, the multi-homogeneous Bézout number  $\text{Béz}(A_1, \dots, A_n; I_1, \dots, I_k)$  is an upper bound on the number of isolated roots of (11) in  $\mathcal{CP}^{a_1} \times \dots \times \mathcal{CP}^{a_k}$ , and thereby an upper bounds the number of isolated finite complex roots of (11). There are a variety of additional methods for computing the multi-homogeneous Bézout number, e.g., [42]. In the particular case where  $A = A_1 = \dots = A_n$ , we denote

$$\text{Béz}(A_1, \dots, A_n; I_1, \dots, I_k) \stackrel{\text{def}}{=} \binom{n}{a_1 \ a_2 \ \dots \ a_k} \prod_{j=1}^k d_j^{a_j}, \quad (12)$$

where  $d_j = d_{ij}$  (equal for each  $i$ ) and the multinomial coefficient

$$\binom{n}{a_1 \ a_2 \ \dots \ a_k} \stackrel{\text{def}}{=} \frac{n!}{a_1! \ a_2! \ \dots \ a_k!}$$

is the coefficient of  $\prod_{j=1}^k \zeta_j^{a_j}$  in  $(\zeta_1 + \dots + \zeta_k)^n$  with  $n = \sum_{j=1}^k a_j$ , as above.

In summary, the multi-homogeneous Bézout number provides a sharper bound on the number of isolated solutions of a system of equations than the usual Bézout number  $\prod_{i=1}^n d_i = d_1 \dots d_n$ . In the famous example of the eigenvalue problem [43], it is known that the Bézout number is  $2^n$ , whereas there exists a structure with multi-homogeneous Bézout number of  $n$ . We hence study the multi-homogeneous structure within the steady state equations of alternating-current circuits.

## 4 The Multi-Homogeneous Structure

Notice that in order to obtain an algebraic system from the steady-state equations (2–5), one needs to reformulate the complex conjugate. In order to do so, one may replace all  $v_n^*$  with independent variables  $u_n$ , and later filter for solutions where  $u_n = v_n^*$  once the complex solutions are obtained. We denote such solution “real”. Let  $G$  be the set of slack generators for which  $|v_n|$  is specified, and assume  $0 \in G$  corresponds to a reference node with phase 0. Notice that the use of variables  $v_n$  and  $u_n$  produces a multi-homogeneous structure with variable groups  $\{v_n\}$  and  $\{u_n\}$ :

$$\begin{aligned} v_n \sum_k Y_{n,k} u_k + u_n \sum_k Y_{n,k}^* v_k &= 2p_n & n \in N \setminus G \\ v_n \sum_k Y_{n,k} u_k - u_n \sum_k Y_{n,k}^* v_k &= 2q_n & n \in N \setminus G \\ v_n u_n &= |v_n|^2 & n \in G - \{0\} \\ v_0 &= |v_0|, \ u_0 = |v_0| & \end{aligned} \quad (13)$$

For example for the two-bus network, we obtain:

$$\begin{aligned} v_1(Y_{1,0}u_0 + Y_{1,1}u_1) + u_1(Y_{1,0}^*v_0 + Y_{1,1}^*v_1) &= 2p_1 \\ v_1(Y_{1,0}u_0 + Y_{1,1}u_1) - u_1(Y_{1,0}^*v_0 + Y_{1,1}^*v_1) &= 2q_1 \\ v_0 = |v_0|, u_0 = |u_0| \end{aligned} \tag{14}$$

Using the algebraic system, one can formulate a number of structural results concerning power flows.

## 5 An Analysis for $s = 1$

For the particular multi-homogeneous structure, which is the partition of the variables into several groups in (13), we can bound the number of isolated solutions:

**Theorem 1.** *With exceptions on a parameter set of measure zero, the alternating-current power flow (13) has a finite number of complex solutions, which is bounded above by:*

$$\binom{2n-2}{n-1} \tag{15}$$

*Proof.* Each equation in the system (13) is linear in the  $V$  variables and also in the  $V^*$  variables, giving rise to a natural multi-homogeneous structure of multi-degree  $(1, 1)$ . Since the slack bus voltage is fixed at a reference value, the system has  $2n - 2$  such equations in  $(n - 1, n - 1)$  variables. By the multi-homogeneous form of Bézout’s Theorem (see e.g. Theorem 8.4.7 in [36]), the total number of solutions in multi-projective space  $\mathcal{CP}^{n-1} \times \mathcal{CP}^{n-1}$  is precisely the stated bound, counting multiplicity. Some subset of these lie on the affine patch  $\mathcal{C}^{n-1} \times \mathcal{C}^{n-1} \subset \mathcal{CP}^{n-1} \times \mathcal{CP}^{n-1}$ , giving the result.  $\square$

Notice that this applies also to some well-known instances of alternating-current optimal power flows (ACOPF). For example, the instances of Lesieutre et al. [22] and Bukhsh et al. [8] have only a single “slack” bus, whose active and reactive powers are not fixed, and hence the result applies. Notice that the exception of measure-zero set is necessary; cf. Example 4.1 of [2].

As we will illustrate in the next section, this bound is tight in some cases. Deciding whether the bound on the number of roots obtained using a particular multi-homogeneous structure is tight for a particular instance is, nevertheless, hard. This can be seen from:

**Theorem 2** (Theorem 1 of Malajovich and Meer [27]). *There does not exist a polynomial time algorithm to approximate the minimal multi-homogeneous Bézout number for a polynomial system (11) up to any fixed factor, unless  $P = NP$ .*

We can, however, show there exists a certain structure among these solutions:

**Corollary 1.** *If there exists a feasible solution of the alternating-current power flow, then the solution has even multiplicity greater or equal to 2 or another solution exists.*

*Proof.* The finite number of solutions to the power flows problem of Theorem 1 is even. Observe that  $(U, \hat{U})$  is a solution of the system (13) if and only if  $(\hat{U}^*, U^*)$  is a solution. This implies that the non-“real” solutions, that is solutions for which  $U^* \neq \hat{U}$ , necessarily come in pairs. It follows that “real” feasible solutions are also even in number, counting multiplicity. The result follows.  $\square$

Note that a solution having multiplicity greater than 1 is a special case that is highly unlikely in a real system. Moreover, it is easily detected, since the Jacobian at a solution is nonsingular if and only if the solution has multiplicity 1.

## 6 Alternating-Current Optimal Power Flow

One may also make the following observations about the alternating-current optimal power flows, i.e., the problem of optimising an objective over the steady state:

**Remark 1.** *For the alternating-current power flows, where powers are fixed at all but the reference bus, whenever there exists a real feasible solution, except for a parameter set of measure zero, one can enumerate all feasible solutions in finite time.*

Indeed: By Theorem 1, we know there exist a finite number of isolated solutions to the system (13). By the homotopy-continuation method of Sommese et al. [36, 6], we can enumerate the roots with probability 1, which allows us to pick the global optimum, trivially. Notice that Bertini, the implementation of the method of Sommese [6], makes it possible to check that all roots are obtained. Notice that the addition of inequalities can be accommodated by filtering the real roots.

Nevertheless, this method is not practical, as there may be too many isolated solutions to enumerate. Generically, this is indeed the case, whenever there are two or more generators with variable output, i.e., buses, whose active and reactive power is not fixed:

**Corollary 2.** *In the alternating-current optimal power flow problem, i.e., with  $s > 1$ , where powers are variable outside of the reference bus and there are no additional inequalities, the complex solution set is empty or positive-dimensional, except for a parameter set of measure zero. When the complex solution set is positive-dimensional, if a smooth real feasible solution exists, then there are infinitely many real feasible solutions.*

*Proof.* For each slack bus after the first, the system has two variables but only one equation. With  $s + 1$  slack buses, the rank of the Jacobian and hence the dimension of the complex solution set will be at least  $s$  by Lemma 13.4.1 in [36]. Furthermore, if a real feasible solution  $U$  exists and the solution set is smooth at this point, then the local dimension of the complex and real solution sets are equal at  $U$ . Therefore, since the complex solution set is positive-dimensional, so is the real set at  $U$ , and so infinitely many real feasible solutions exist.  $\square$

Although there are a variety of methods for studying positive-dimensional systems, including the enumeration of a point within each connected component [5, 33] and studying the critical points of the restriction to the variety of the distance function to such points [1], we suggest the method of moments [20, 13] may be more suitable for studying the feasible set of alternating-current optimal power flows. It has been shown recently [13] that it allows for very small errors on systems in dimension of over five thousand.

As is often the case in engineering applications, one may also be interested in the distance of a point to the set of feasible solutions. Again, by considering our algebraisation, one can bound the probability this distance is small using the theorem of Lotz [25] on the zero-set  $V$  of multivariate polynomials. This could be seen as the converse of the results of [24].



Method	No. $ N $ of buses											
	3	4	5	6	7	8	9	10	11	12	13	14
Bézout's upper bound	16	64	256	1024	4096	16384	65536	262144	1048576	4194304	16777216	67108864
A BKK-based upper bound	8	40	192	864	3712	15488	63488	257536	1038336	4171776	16728064	67002368
Theorem 1	6	20	70	252	924	3432	12870	48620	184756	705432	2704156	10400600
Generic lower bound	6	20	70	252	924	3432	12870	48620	184756	705432	2704156	10400600

Table 1: The maximum number of steady states in a circuit with a fixed number of buses.

Instance	Source	No. $ N $ of buses	No. $ E $ of branches	Treewidth $\text{tw}(P)$	No. $ X $ of solutions	$\min_{x \in X}(\text{cost}(x))$	No. of min. wrt. cost	$\text{avg}_{x \in X}(\text{cost}(x))$	$\max_{x \in X}(\text{cost}(x))$	$\min_{x \in X}(\text{loss}(x))$	No. of min. wrt. loss	$\text{avg}_{x \in X}(\text{loss}(x))$	$\max_{x \in X}(\text{loss}(x))$
case2w	Bukhsh et al. [8]	2	1	1	2	8.42	1	9.04	9.66	0.71	1	1.02	1.33
case3KW	Klos and Wojcicka [18]	3	3	2	6	-0.0	1	1250.0	1500.0	-0.0	1	-0.0	0.0
case3LL	Lavaei and Low [21]	3	3	2	2	1502.07	1	1502.19	1502.31	0.22	1	0.34	0.46
case3w	Bukhsh et al. [8]	3	2	1	2	5.88	1	5.94	6.01	0.55	1	0.58	0.61
case4	McCoy et al.	4	3	1	4	1502.51	1	1505.19	1507.88	0.11	1	2.79	5.48
case4ac	McCoy et al.	4	4	2	6	2005.25	2	3337.5	4005.51	0.01	1	2.44	3.78
case4cyc	Bukhsh et al. [8]	4	4	2	4	3001.61	1	3003.56	3005.52	0.01	1	1.96	3.92
case4gs	Grainger and Stevenson [45]	4	4	2	10	2316.37	1	3681.07	4595.8	0.37	1	27.24	39.72
case5w	Lesieutre et al. [22]	5	6	2	2	2003.57	1	2004.6	2005.63	0.47	1	1.5	2.53
case6ac	McCoy et al.	6	6	2	23	4005.44	1	5574.23	6013.26	0.02	1	6.26	10.51
case6ac2	McCoy et al.	6	6	2	22	4005.32	1	5554.03	6012.82	0.02	1	5.97	10.21
case6b	McCoy et al.	6	6	2	30	3005.7	5	3606.3	4508.28	0.01	1	3.9	5.88
case6cyc	Bukhsh et al. [8]	6	6	2	30	3005.7	4	3606.3	4508.28	0.01	1	3.9	5.88
case6cyc2	McCoy et al.	6	6	2	12	1506.87	1	3131.55	4508.24	0.03	1	4.15	6.56
case6cyc3	McCoy et al.	6	6	2	12	4502.42	1	4505.95	4508.01	0.02	1	3.55	5.61
case7	McCoy et al.	7	7	2	2	1667.31	1	1668.36	1669.41	0.08	1	0.26	0.45
case8cyc	Bukhsh et al. [8]	8	8	2	60	6506.11	1	8557.72	9511.04	0.01	1	4.52	7.84
case9	Chow [45]	9	9	2	16	3504.44	1	5692.02	6505.02	0.03	1	1.37	1.87
case9g	McCoy et al.	9	9	2	2	2604.41	1	4103.36	5602.31	0.08	1	0.26	0.45
caseK4	McCoy et al.	4	6	3	8	2008.62	1	3006.32	4005.51	0.01	1	4.6	7.03
caseK4sym	McCoy et al.	4	6	3	6	2009.28	2	3339.29	4005.91	0.01	1	3.96	7.28
caseK6	McCoy et al.	6	15	5	36	2018.2	1	5075.47	6030.0	0.02	1	17.16	27.25
caseK6b	McCoy et al.	6	15	5	40	6002.79	1	6017.01	6019.41	0.01	1	14.23	16.62
caseK6sym	McCoy et al.	6	15	5	48	6003.01	1	6017.75	6019.86	0.01	1	14.75	16.86

Table 2: Properties of the instances tested.

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variable_group V0, V1; variable_group U0, U1;
I0 = V0*Yv0_0 + V1*Yv0_1; I1 = V0*Yv1_0 + V1*Yv1_1;
J0 = U0*Yu0_0 + U1*Yu0_1; J1 = U0*Yu1_0 + U1*Yu1_1;
fv0 = V0 - 1.0; fv1 = I1*U1 + J1*V1 + 7.0;
fu0 = U0 - 1.0; fu1 = -I1*U1 + J1*V1 - 7.0*I;

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Figure 1: A Bertini encoding of ACPF on the two-bus instance of Bukhsh et al. 2012, where the impedance of a single branch is  $0.04 + 0.2i$ .

## 7 Computational Illustrations

In order to illustrate Theorem 1, we first present the maximum number of steady states in a circuit with a fixed number of buses in Table , and compare it to the values of our upper bound, Bézout-based upper bound, and the BKK-based upper bound [10, 31]. Notice that the maximum number of steady states in a circuit with a fixed number of buses is achieved when  $(N, E)$  is a clique. Notice further that the generic lower bound, obtained as the number of solutions found by tracing the paths, matches the upper bound of Theorem 1 throughout Table 7.

To illustrate Proposition 1, we have enumerated the steady states using Bertini, a versatile package for homotopy-continuation methods by Sommese et al. [36]. See Figure 1 for an example of Bertini input corresponding to the example above (14), with constants  $Yu_{i-j}$  representing  $Y_{i,j}$  and  $Yv_{i-j}$  representing  $Y_{i,j}^*$ . The results are summarised in Table .

To illustrate Theorem 1 further, we present the values of our upper bound on a collection of instances widely known in the power systems community. The instances are mostly available from the Test Case Archive of Optimal Power Flow (OPF) Problems with Local Optima of Bukhsh<sup>1</sup>, while some have appeared in well-known papers, e.g. [18], and some are available in recent distributions of Matpower [45], a well-known benchmark. In particular, we present the numbers of distinct roots of the instances. In all cases, where Theorem 1 applies, the number of solutions found by tracing the paths matches the upper bound of Theorem 1, certifying the completeness. In other cases, one could rely on Bertini certificates of completeness of the search.

Empirically, we observe there exists a unique global optimum in all these instances tested with respect to the  $L_1$ -loss objective. For the generation cost objective, however, there are a number of instances (case4ac, caseK4sym, case6b, case6cyc), where the global optima are not unique. The case of caseK4sym is a particularly good illustration, where the symmetry between two generators and two demand nodes in a complete graph results in multiple global optima.

In order to provide material for further study of structural properties, we present tree-width of the instances in Table 7 in column  $tw(N)$ . Notice that for the well-known small instances, tree-width is 1 or 2, e.g., 1 for the instance

<sup>1</sup><http://www.maths.ed.ac.uk/OptEnergy/LocalOpt/>, accessed November 30th, 2014.

in Figure 1, and 2 for the instance of Lesieutre et al. As the instances grow, however, this need not be the case: Kloks [17] shows treewidth is not bounded even in sparse random graphs, with high probability. In complete graphs, such as caseK4sym with tree-width 3 above, tree-width grows linearly in the number of vertices.

## 8 Related Work

There is a long history of study of the number and structure of solutions of power flows [37, 4, 3, 2]. [37] considered the Bézout bound. [10, 31] considered a bound based on the work of Bernstein [7] and Kushnirenko [19]. [2, 4] derived the same expression as in Theorem 1 using intersection theory, but in the lossless AC model. They highlight that the number of solutions in an alternating-current model with losses is an important open problem. We note that Theorem 1 subsumes Theorem 4.1 of [2] as a special case. Finally, we note that [18] present a lower bound without a proof and recent papers [39, 44] bound certain distinguished solutions, but not all solutions; cf. [28].

There is also a long history of applications of homotopy-continuation methods in power systems [34, 14, 26, 23], although often, e.g. in [26], the set-up of the homotopy restricted the methods to a heuristic, which could not enumerate all the solutions of the power flow [30]. Recently, these have attracted much interest [31, 29, 9] following the work of Trias [39, 40, 41]. See [28] for an overview.

## 9 Conclusions

We hope that the structural results provided will aid the development of faster solvers for the related non-linear problems [15]. Arguably, one could:

- By using Theorem 1 in the construction of start systems for homotopy-continuation methods [43], allow for larger zero-dimensional systems to be studied.
- Extend Corollary 2 to finding at least one point in each connected component [33, 1].
- Extend the homotopy-continuation methods to consider inequalities within the tracing, rather than only in the filtering phase, which could improve their computational performance considerably.
- Develop methods for the optimal power flow problem, whose complexity would be superpolynomial only in the tree-width and the number of buses.

The latter two may be some of the most important challenges within the analysis of circuits and systems.

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